By special arrangement, the paper by Miss McKelden appears as a supplement to the present number of the MONTHLY. It is the policy of the Editors to devote the regularly available space to elementary papers of general interest, and not to technical papers, however great their value.

THE AMERICAN MATHEMATICAL MONTHLY.

Entered at the Post-office at Springfield, Missouri, as second-class matter.

VOL. XIII. JUNE-JULY, 1906. NOS. 6-7.

GROUPS OF ORDER 2^m THAT CONTAIN CYCLIC SUBGROUPS OF ORDER 2^{m-3}.

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INTRODUCTION.

A determination of groups of order p^3, and p^4 has been given by Hölder;* another determination of groups of orders p^3 and p^4, by Young.† The distinct types of groups of orders p^2, p^3, p^4 have been constructed and tabulated by Burnside.‡ Types of groups of order p^5, in addition to those of order p^3, p^4, which were considered first to illustrate the method of treatment, have been determined by Bagnera;§ and types of groups of order p^6 have been determined by Potron.¶ The number of groups of order p^m, which contain self-conjugate cyclic subgroups of orders p^{m-1} and p^{m-2}, respectively, has been discussed by Burnside;‖ the number of groups of order p^m that contain cyclic non-self-conjugate subgroups of order p^{m-2} has been determined by Miller;** and the groups of order p^m, which contain cyclic subgroups of order p^{m-3} (p odd prime) have been determined by Neikirk.††

Mathematische Annalen, Vol. 43 (1893), pp. 301—412.

†*American Journal of Mathematics*, Vol. 15 (1893), pp. 124—178.

‡*Theory of Groups of a Finite Order*, pp. 81—89.

§*Annali di Mathematica*, Vol. 3 (1898), pp. 137—228; 263—275.

¶ *These* (1904), Gauthier Villars, Paris.

‖ *Loc. Cit.* pp. 75—81. One of the groups has been omitted by Burnside and XI~XII, p. 81. See "A Note on Groups of Order 2^m, which contain Self-Conjugate Subgroups of Order 2^{m-2}," Hallett, *Science*, New Series, Vol. 21, No. 527, Feb. 3, 1905.

***Transactions American Mathematical Society*, Vol. 2 (1901), p. 259, and Vol. 3 (1902), p. 383.

††Publications of The University of Pennsylvania, Mathematics, No. 3. *Transactions of American Mathematical Society*, Vol. 6 (1905), No. 3.

The object of this paper is to treat in detail the construction of groups of order 2^m $(m>6)$ that contain cyclic subgroups of order 2^{m-3}, and to give a complete tabulation of such groups.

The treatment is based on the following division of the groups, where P is an operator of G_m of maximum order:

I. P is of order 2^m; II. P is of order 2^{m-1}; III. P is of order 2^{m-2}; IV. P is of order 2^{m-3}. The group G_m contains a series of subgroups G_{m-1}, G_{m-2}, ..., G_{m-r}, of order 2^{m-1}, 2^{m-2}, ..., 2^{m-r}, containing any G_{m-r}; and each one is self-conjugate in the one preceding.*

I, II, III. GROUPS CONTAINING P, OF ORDER 2^m, 2^{m-1}, OR 2^{m-2}.

These groups have all been determined and tabulated by Miller† and Burnside,‡ and the thirty-three types are given here for reference.

I. Abelian group of the type (m).

II. $P^{2^{m-1}}=1$.

$Q^{-1}PQ=P^{\omega+2^{m-2}\kappa}$, $Q^2=P^{2^{m-2}\lambda_1}$; $\omega=\pm 1$, $\kappa=0, 1$, $\lambda_1=0$; $\omega=-1$, $\kappa=0$, $\lambda_1=1$.

III. $P^{2^{m-2}}=1$.

$Q^{-1}PQ=P^{\omega+2^{m-4}\beta_1}$, $Q^4=P^{2^{m-3}\lambda_1}$; $\omega=\pm 1$, $\beta_1=0, 1, 2$, $\lambda=0$; $\omega=-1$, $\beta_1=0$, $\lambda_1=1$.

$Q^{-1}PQ=Q^2P^{1-2^{m-4}\kappa}$, $Q^4=1$, $Q^{-2}PQ^2=P^{1+2^{m-3}\kappa}$, $\kappa=0, 1$.

$Q^{-1}PQ=Q^2P^{-1+2^{m-3}\beta_2}$, $Q^4=1$, $Q^{-2}PQ^2=P^{1+2^{m-3}\kappa}$, $\beta_2=0, 1$, $\kappa=0, 1$.

$Q^{-1}PQ=Q^2P^{-1}$, $Q^4=P^{2^{m-3}}$, $Q^{-2}PQ^2=P^{1+2^{m-3}\kappa}$, $\kappa=0, 1$.

$Q^{-1}PQ=Q^2P^{-1+2^{m-4}\beta_2}$, $Q^4=P^4$, $\beta_2=0, 1$, $\kappa=0$; $\beta_2=0$, $\kappa=1$.

$Q^{-1}PQ=P^{1+2^{m-3}\kappa}$, $R^{-1}PR=P^{\omega_1+2^{m-3}\beta_1}$, $R^{-1}QR=QP^{2^{m-3}b_1}$, $Q^2=1$, $R^2=1$,

$\omega_1=\pm 1$, $\kappa=0$; $\beta_1=0, 1$, $b_1=0$; $\beta_1=0$, $b_1=1$; $\omega_1=-1$, $\kappa=\beta_1=1$, $b_1=0, 1$.

$Q^{-1}PQ=P$, $R^{-1}PR=P^{-1}$, $R^{-1}QR=Q$, $Q^2=1$, $R^2=P^{2^{m-3}}$.

IV. GROUPS CONTAINING P, OF ORDER 2^{m-3}.

The eighth power of every operator is in $\{P\}$. The groups may be divided into three classes.

Class 1. An operator Q of G_m may be so chosen that Q^4 is not contained in $\{P\}$.

Class 2. The fourth power of every operator is in $\{P\}$ and there is an operator Q, of G_m, such that Q^2 is not in $\{P\}$.

Class 3. The second power of every operator is contained in $\{P\}$.

These classes will be treated in order in Parts 1, 2, and 3, respectively.

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*Burnside, *Loc. Cit.*, Art. 55, p. 65.

†Miller, *Transactions American Mathematical Society*, Vol. 2 (1901), p. 259, and Vol. 3 (1902), p, 383.

‡Burnside, *Loc. Cit.*, pp. 75—81.

PART 1. Q^4 IS NOT IN $\{P\}$.

In this case $Q^8=P^{8\lambda}$, and G_m is generated by P and Q, since there are 2^m distinct operators of the form $Q^\alpha P^\beta$ ($\alpha=0\ldots7$, $\beta=0\ldots2^{m-3}-1$). Also $G_{m-3}=\{P\}$, and G_{m-2} is generated by P and some other operator of G_m, $Q^\alpha P^\beta$. Hence G_{m-2} contains Q^α. So $G_{m-2}=\{P, Q^\alpha\}=\{P, Q^4\}$. Likewise $G_{m-1}=\{P, Q^2\}$.

In G_{m-2}, $Q^{-4}PQ^4=P^{\omega+2^{m-4}\kappa}$; in G_{m-1}, $Q^{-2}PQ^2=Q^{4a}P^\beta$, and in G_m, $Q^{-1}PQ=Q^{2\alpha}P^b$. Three cases arise: (A) $a=\alpha=0$; (B) $a=0$, $\alpha=1, 2, 3$; (C) $a=1$, $\alpha=1, 2, 3$. We shall subdivide these cases where necessary, using A_1, A_2, ... to distinguish the 1st, 2nd, ... subcases of A, etc.

(A) $a=\alpha=0$. Here $\{P\}$ is self-conjugate in G_m.

$$Q^{-1}PQ=P^b=P^{\omega_1+2^{m-6}b_1}\ (\omega_1=\pm1,\ b_1=0\ldots7)\ldots(1).$$

Let $b_1=2^nb_2$, where b_2 is odd and $n=0\ldots3$. In general, $(R^zQ^yP^xR^v\ldots)^s$ will be represented by $[z, y, x, v\ldots]^s$; also $\dfrac{1+(-1)^{s-1}}{2}$ by θ_s, and $\dfrac{1+(-1)^s}{2}$ by ϕ_s. From (1),

$$[0, -y, x, 0, y]=[0, 0, x\,(\omega_1+2^{m-6+n}b_2)^y]\ldots(2).$$

$$[0, y, x]^2=[0, 2y, 2x\{\frac{1+(\omega)^y}{2}+(\omega)^{y-1}2^{m-7+n}b_2y\}]\ldots(3).$$

Let $Q'=QP^x$; then $Q'^8=1$, if x be chosen to satisfy

$$\lambda+x(1+\omega_1)/2+2^{m-7+n}b_2x\equiv0\ (\mathrm{mod}\ 2^{m-6})\ldots(4),$$

where for $\omega_1=-1$, $\lambda=2^{m-7}\lambda_1$. This is always possible for $\omega_1=1$, except when $m=7$; b_1 and λ odd, when $Q^8=P^8$; and for $\omega_1=-1$, when b_1 is odd. For $\omega_1=-1$, and b_1 even, $Q'^8=P^{2^{m-4}\lambda_1}$; $\lambda_1=0, 1$. In this last case,

$$[0, y, x]^s=[0, sy, x\,(s\phi_y+\theta_{sy})+2^{m-7+n}b_2xy\{s-s^2\phi_y-(2s-1)\theta_{sy}\}]\ldots(5),$$

and the groups, where $n=2$, correspond to those where $n=3$, for $\lambda_1=1$. The correspondence is given by $C=\begin{bmatrix}Q^4P, & Q\\ P, & Q\end{bmatrix}$.

Let $Q'=Q^y$; then $Q'^{-1}PQ'=P^{\omega_1+2^{m-6+n}}$, if y be chosen to satisfy

$$b_2y+\frac{y\,(y-1)}{2}2^{m-6+n}b_2^2+\ldots\equiv1\ (\mathrm{mod}\ 2^{3-n})\ldots(6),$$

which choice is always possible. So in (A) there are ten types. They are given by the following defining relations:

$Q^{-1}PQ=P^{\omega_1+2^{m-6+n}}$, $Q^8=P^{2^{m-4}\lambda_1}$, $P^{2^{m-3}}=1$; $\lambda_1=0$; $\omega_1=\pm1$, $n=1, 2, 3$; $m>7$, $\omega_1=\pm1$, $n=0$. $m=7$, $\omega_1=-1$, $n=0$; $\lambda_1=1$, $\omega_1=-1$, $n=1, 3$.

$Q^{-1}PQ=P^{1+2}$, $Q^8=P^8$, $P^{2^4}=1$.

(B) $a=0$, $a=1, 2, 3$.

$$Q^{-1}PQ=Q^{2a}P^{b}\ldots(1),$$

$$Q^{-2}PQ^{2}=P^{\omega_1+2^{m-5}\beta_1}\ (\omega_1=\pm 1,\ \beta_1=0\ldots 3)\ldots(2).$$

From (2), $[0, -2y_1, x, 0, 2y_1]=[0, 0, x(\omega_1+2^{m-5}\beta_1)^{y_1}]\ldots(3)$. Hence

$$[0, 2y_1, x]^{s}=[0, 2sy_1, \theta_s x\{1+\omega_1 2^{m-5}\beta_1 y_1(s-\theta_s)\}+(s-\theta_s)\ \{\frac{1+(\omega_1)^{y_1}}{2} + (\omega_1)^{y_1-1}2^{m-6}\beta_1 y_1]x]\ldots(4).$$

Transform (1) by Q and place equal to (2); raise to the power 8λ. There result $b=1+2b_1$, and the congruences

$$a(1+b_1)\equiv 0 \pmod 2\ldots(5);$$

$$\lambda[4a\lambda-1+2b_1+(\omega_1)^{a}(1+2b_1)]\equiv 0 \pmod{2^{m-5}}\ldots(6),$$

$$4a\lambda(1+b_1)+(1+2b_1)[1+b_1(1+(\omega_1)^{a})]+2^{m-5}\beta_1[(\omega_1)^{a-1}ab_1+1] \equiv \omega_1 \pmod{2^{m-3}}\ldots(7).$$

For $\omega_1=-1$, a and b are odd by (8). Hence P is of an order lower than 2^{m-3}, and there are no groups in this case.

For $\omega_1=1$, from (1) and (4),

$$[0, -y, x, 0, y]=[0, 2ax\theta_y, x\{1+2^{m-6}\beta_1(y-\theta_y)\}\{1+2\theta_y[b_1+2^{m-7}a\beta_1(1+2b_1)]\} +2^{m-6}a\beta_1\theta_y\{2\lambda x(y-1)+\theta_x(1+2b_1)(2x-3)\}]\ldots(8),$$

and $[0, y, x]^{2}=[0, 2(y+ax\theta_y), x+x\{1+2^{m-6}\beta_1(y-\theta_y)\}\{1+2\theta_y[b_1+2^{m-7}a\beta_1 \times(1+2b_1)]\}+2^{m-6}a\beta_1\theta_y\{2\lambda x(y-1)+\theta_x(1+2b_1)(2x-3)\}]\ldots(9)$.

(B_1) $a=2$.

(a) $b_1=2^{m-7}b_2$; (b) $b_1=-1+2^{m-7}b_2$; $(b_1=0\ldots 7)$. Let $Q'=QP^{-\lambda}$. Then, for (a), $Q'^{8}=1$; for (b), $Q'^{8}=P^{2^{m-4}\lambda_1}$ $(\lambda_1=0, 1)$. $(b)\sim(A)$ with $C=\begin{bmatrix} Q^2P, & Q \\ P, & Q \end{bmatrix}$. $(a)\sim(a_1)$, (a_2), where $b_2=\beta_1=0, 2$, respectively; and (a_3) where $b_2=\beta_1=1$, for $m>8$; and $b_2=3$, $\beta_1=1$, for $m=8$. $C=\begin{bmatrix} P, & Q^{y'}Q^{2^{m-7}x_1'} \\ P, & Q \end{bmatrix}$, where $y'=1+2y'_1$ and y'_1 and x'_1 are determined to satisfy the congruences $\beta_1(1+2y'_1)\equiv\beta'_1 \pmod 4$; $\lambda_1+x'_1\equiv\lambda'_1 \pmod 2$; and $2y'_1(\beta_1-2\lambda_1)-2x'_1\equiv b'_2-b_2 \pmod 8$. Of these (a_3) for $m=8$, and (a_2) for $m=7\sim(A)$ with $C=\begin{bmatrix} QP, & QP^4 \\ P, & Q \end{bmatrix}$ and $\begin{bmatrix} QP, & QP^2 \\ P, & Q \end{bmatrix}$, respectively. Hence there are three types for $m>8$, two for $m=8$, and one for $m=7$.

(B_2) $a=1, 3$, $m>7$. From (5), $b_1=1+2b_2$. Let $Q'=QP^{-\lambda}$ for λ even. Then $Q'^{8}=P^{2^{m-4}\lambda_1}$ $(\lambda_1=0, 1)$. Let $Q'=Q^{y}$ (y odd), for λ odd. Then $Q'^{8}=P^{8}$, if y be determined to satisfy $\lambda y\equiv 1 \pmod{2^{m-6}}$.

(a) For $\lambda=2^{m-7}\lambda_1$, from (7) $b_2=-1+2^{m-7}b_3$ ($b_3=0\ldots3$). (a)$\sim$the seven types $a=1$; $b_3=\beta_1=0$, 1, 2, $\lambda_1=0$, 1; $b_3=2$, $\beta_1=0$, $\lambda=0$; with $C=\begin{bmatrix} Q^{2y_1}P^x, & QP^{2x_1'} \\ P, & Q \end{bmatrix}$, where $x=1+2x_1$, and the variables satisfy $a_1-a_1'\equiv x_1+y_1+x_1'$ (mod 2); $\beta_1(1+2x_1')\equiv\beta_1'$ (mod 4); and $\lambda_1\{a_1-a_1'+x_1+y_1-x_1'+2[a_1(x_1-x_1')-a_1'x_1']\}-\beta_1\{1-x_1-x_1'-b_3'y_1+2[a_1(x_1+x_1'+1)+x_1(1+y_1)+a_1'x_1']\}+(b_3-b_3')(1+2x_1)+2b_3x_1'\equiv0$ (mod 4).

(b) For $\lambda=1$, from (6), $b_2=-1-a_1+2^{m-8}b_3$, ($b_3=0\ldots7$; $a=1+2a_1$). (b)$\sim$the seven types $a=1$; $b_3=0$, $\beta_1=0$, 2; $a=3$, $b_3=\beta_1=0$, 1, 2; $m>8$, $a=1$; $m=8$, $a=3$; $b_3=1$, $\beta_1=0, 2$; with $C=\begin{bmatrix} P, & QP^{2^{m-6}x_1'} \\ P, & Q \end{bmatrix}$, where x_1' satisfies $b_3'-b_3+2ax_1'\equiv0$ (mod 8),* except for β_1 odd, where $C=\begin{bmatrix} P^{\beta_1}, & QP^{2(\beta_1'-1)+2^{m-6}x_1'} \\ P, & Q \end{bmatrix}$ where x_1' satisfies $b_3'\beta_1'-b_3+2ax_1'\equiv0$ (mod 8).

(B_3) $a=1, 3$; $m=7$. (B_3), ($a=3$)$\sim$(B_3), ($a=1$); with $C=\begin{bmatrix} P^3, & Q \\ P, & Q \end{bmatrix}$.

(a) $\lambda=0$. There are four types $b_2=1$, 3, $\beta_1=0$, 2.

(b) $\lambda=1$. (b)$\sim$the five types $b_2=0$, 1, $\beta_1=0$, 2, and $b_2=0$, $\beta_1=1$; with $C=\begin{bmatrix} P, & QP^{4x_1'} \\ P, & Q \end{bmatrix}$ where x_1' satisfies $b_2'-b_2+2ax_1'\equiv0$ (mod 4), except for β_1 odd, where $C=\begin{bmatrix} P^x, & Q^{y'}P^{2x_1'} \\ P, & Q \end{bmatrix}$, where x and y' are odd and the variables satisfy $x_1+x_1y_1'+x_1'y_1'-b_3'\equiv0$ (mod 2); $\beta_1'\equiv y'+2x_1'$ (mod 4); and $x_1+x_1'+y_1'\equiv0$ (mod 2).

Hence in (B) there are seventeen types for $m>8$, sixteen for $m=8$, ten for $m=7$, viz:

$Q^{-2}PQ^2=P^{1+2^{m-5}\beta_1}$, $P^{2^{m-3}}=1$.

$Q^{-1}PQ=Q^4P^{1+2^{m-6}b_2}$, $Q^8=1$; $m>8$; $b_2=\beta_1=0$, 1, 2; $m=8$, $b_2=\beta_1=0$, 2; $m=7$, $b_2=\beta_1=0$.

$Q^{-1}PQ=Q^2P^{-1+2^{m-4}b_3}$, $Q^8=1$; $m>7$, $b_3=1$, $\beta_1=0$; $m=7$, $b_3=0$, 1, $\beta_1=0, 2$.

$Q^{-1}PQ=Q^2P^{-1+2^{m-5}b_3}$, $Q^8=P^{2^{m-4}\lambda_1}$; $m>7$; $b_3=\beta_1=0$, 1, 2; $\lambda_1=0, 1$; $m=7$, $b_3=1$, $\beta_1=0$, 1, 2, $\lambda_1=1$.

$Q^{-1}PQ=Q^{2(1+2a_1)}P^{-1-4a_1+2^{m-6}b_3}$, $Q^8=P^8$; $a_1=b_3=0$, $\beta_1=0$, 2; $m>8$, $a_1=0$, $m=8$, $a_1=1$, $b_3=1$, $\beta_1=0$, 2; $m=7$, $a_1=1$, $b_3=\beta_3=0$, 1, 2.

(C) $\alpha=1$, $a=1$, 2, 3.

$$Q^{-1}PQ=Q^{2a}P^b\ldots(1),$$

$$Q^{-2}PQ^2=Q^4P^\beta\ldots(2),$$

$$Q^{-4}PQ^4=P^{\omega+2^{m-4}\kappa}\ldots(3).$$

*For $m=8$, b_3 odd and $a=3$, x_1' satisfies $b_3'-b_3+2x_1'\equiv0$ (mod 8).

From (3), $[0, -4y_1, x, 0, 4y_1]=[0, 0, x(\omega+2^{m-4}\kappa)^{y_1}]\ldots(4)$, and

$$[0, 4y_1, x]^s=[0, 4sy_1, (s-\theta_s)\{\frac{1+(\omega+2^{m-4}\kappa)^{y_1}}{2}\}x+\theta_s x]\ldots(5).$$

Now $\omega=1$; for when $\omega=-1$, (2) raised to the second power shows that P is of an order lower than 2^{m-3}. Also a is odd; for if $a=2$, (1) transformed by Q and placed equal to (2) shows that b is even, which makes P of an order lower than 2^{m-3}. Next transform (2) by Q^2; $\beta=1+2\beta_1$. From (5) and (2),

$$[0, -2y_1, x, 0, 2y_1]=[0, 4x\theta_{y_1}, x(1+2\beta_1\theta_{y_1})+2^{m-5}\kappa(xy_1-\theta_{xy_1})]\ldots(6).$$

Hence,

$$[0, 2y_1, x]^s=[0, 2sy_1+2(s-\theta_s)x\theta_{y_1}, sx+2(s-\theta_s)\{\beta_2 x\theta_{y_1}+2^{m-7}\kappa(xy_1-\theta_{xy_1})\}].(7).$$

Square (1) by (7); $b=1+2b_1$, $\beta_1=2\beta_2$. From (1) and (7),

$$[0, -y, x, 0, y]=[0, 4x\theta_{y_1}+2\theta_y\{ax+b(x-\theta_x)\}, 16\lambda\beta_2 x\theta_{yy_1}(1+a_1+b_1)+\{x(1+4\beta_2\theta_{y_1})+2^{m-5}\kappa(xy_1-\theta_{xy_1})\}\{1+2\theta_y(b_1+\beta_2+2b_1\beta_2)\}+2\theta_y\{-\theta_x\beta_2(1+2b_1)+2^{m-6}\kappa(a_1+b_1)(x-\theta_x)\}]\ldots(8).$$

Hence,

$$[0, y, x]^2=[0, 4x\theta_{y_1}+2\{y+\theta_y[ax+b(x-\theta_x)]\}, 16\lambda\beta_2\theta_{yy_1}x(1+a_1+b_1)+x+x(1+4\beta_2\theta_{y_1})\{1+2\theta_y(b_1+\beta_2+2\beta_2 b_1)\}+2^{m-5}\kappa(xy_1-\theta_{xy_1})+2\theta_y\{2^{m-6}\kappa(a_1+b_1)(x-\theta_x)-\beta_2\theta_x(1+2b_1)\}]\ldots(9).$$

Now transform (1) and (2) by Q^2 and Q, respectively; then raise them to the power 8λ. There result:

$$\beta_2=-\lambda+2^{m-7}\beta_3, (\beta_0=0\ldots3); \beta_3\equiv\kappa \pmod 2\ldots(10);$$

$$\lambda\{\lambda(1+2a_1)+b_1\}\equiv0 \pmod{2^{m-7}}\ldots(11), \text{ and}$$

$$(1+b_1)[\lambda(1+2a_1)+b_1]+2^{m-7}[\beta_3(b_1-1)+2\kappa b_1 a_1]\equiv0 \pmod{2^{m-5}}\ldots(12).$$

(C_1) $b=2b_2$, $m>7$. From (11), $\lambda=2\lambda_1$. Let $Q'=QP^{2x_1}$; then $Q'^8=1$, where x_1 satisfies $\lambda_1(1+2ax_1)+x_1(1+2b_2)\equiv0 \pmod{2^{m-5}}$. From (12), $b_1=2^{m-7}b_3$ ($b_3=0\ldots7$). Thus (C_1)$\sim$the four types $a=1, 3$; $\kappa=\beta_3=b_3=0, 1$; with $C=\begin{bmatrix}P, & QP^{2^{m-6}x_1'}\\ P, & Q\end{bmatrix}$, where x_1' satisfies $\beta_3-\beta_3'+2x_1'\equiv0 \pmod 4$, and $b_3-b_3'\equiv2a'x_1' \pmod 8$.

(C_2) $b=1+2b_2$, $m>7$.

(a) For λ even, from (12), $b_1=-1+2^{m-6}b_3$ ($b_3=0\ldots3$). From (11), $\lambda=2^{m-7}\lambda_2$ ($\lambda_2=0, 1$). Hence (a)$\sim$the fourteen types $\lambda_2=0, 1$; $\kappa=\beta_3=b_3=0, 1$, $a=1$; $\kappa=\beta_3=0, 1$, $b_3=0$, $a=3$; $\lambda_2=\kappa=0$, $\beta_3=0, 2$, $b_3=2$, $a=1, 3$; $\lambda_2=\kappa=1$, $\beta_3=3$; $b_3=a=1$; $b_3=0$, $a=3$. The isomorphism is given by $C=\begin{bmatrix}Q^{2y_1}P^x, & QP^{2x_1'}\\ P, & Q\end{bmatrix}$ where $x=1+2x_1$, and the variables satisfy $(\beta_3-\beta_3')x+2[\beta_3x_1'+\lambda_2x_1'+\kappa x_1]\equiv0 \pmod 4$, and $\lambda_2\{2a_1x_1+x_1+y_1-x_1'\}-\kappa\{2(a_1-1)(x_1+x_1')-(y_1-\theta_{y_1})\}-\beta_3\{x_1-a'x_1'-\theta_{y_1}\}-2b_3x_1'+(b_3-b_3')x\equiv0 \pmod 4$.

(b) For λ odd, from (11) $b_1=-1-2a_1+2^{m-7}b_3$. Let $Q'=Q^y$ (y odd). Then $Q'^8=P^8$ where y satisfies $\lambda_y\equiv 1 \pmod{2^{m-6}}$. Thus ($b$)$\sim$the seven types $\kappa=\beta_3=0$; $b_3=0$, 2, $a=1$, 3; $b_3=1$, $a=1$;* $\kappa=\beta_3=1$, $b_3=1$, 3; with $C=\begin{bmatrix} P, & QP^{2^{m-6}x_1'} \\ P, & Q \end{bmatrix}$, where x_1' satisfies $\beta_3-\beta_3'+2x_1'\equiv 0 \pmod 4$, and $b_3-b_3'-2x_1'(1+2a_1)\equiv 0 \pmod 8$;† except for $a=1$, $\kappa=0$; $\beta_3=0$, $b_3=3+4b_4$;‡ $\beta_3=2$, $b_3=1+4b_4$, $b_4=0$, 1; where $C=\begin{bmatrix} Q^2P^x, & Q \\ P, & Q \end{bmatrix}$, with $x=1+2(2^{m-6}-1)$, -1, $1+2(\pm 2^{m-7}-1)$, respectively.

(C_3) $m=7$. From (10), $\beta_2=-\lambda-\kappa+2\beta_3$.

(a) $b_1=2b_2$. Here (a)$\sim$the four types, $\beta_3=0$; $a=1$, 3, $\kappa=b_2=\lambda=0$; $\kappa=\lambda=1$, $a=1$, $b_2=1$, $a=3$, $b_2=0$, with $C=\begin{bmatrix} P, & QP^{2x_1'} \\ P, & Q \end{bmatrix}$, where x_1' satisfies $\beta_3+\beta_3'\equiv x_1' \pmod 2$ and $b_2-b_2'-x_1'(\lambda-\kappa+a+2\beta_3+2b_2)\equiv 0 \pmod 4$.

(b) $b_1=1+2b_2$. Here (b)$\sim$the fourteen types, $a=1$, 3, $\kappa=0$; $\lambda=0$, $\beta_3=0$, $b_2=1$, 3; $\beta_3=1$, $b_2=1$; $\lambda=1$, $\beta_3=0$, $b_3=0$, 1; $\kappa=1$, $\beta_3=0$; $b_2=0$; $a=3$, $\lambda=0$; $a=1$, $\lambda=1$; $b_2=1$, 3, $a=3$, $\lambda=1$; with $C=\begin{bmatrix} P^x, & QP^{2x_1'} \\ P, & Q \end{bmatrix}$, where $x=1+2x_1$ and the variables satisfy $\beta_3+\beta_3'+x_1'(\lambda-\kappa)-\lambda x_1\equiv 0 \pmod 2$, and $(b_2-b_2')x+(\lambda-\kappa)(ax_1-x_1')-2\beta_3(x_1+x_1')-2x_1'(1+b_2)\equiv 0 \pmod 4$. Hence in ($C$) there are twenty-five types for $m>7$; eighteen for $m=7$. The defining relations of these types are the following:

$Q^{-4}PQ^4=P^{1+2^{m-4}\kappa}$, $P^{2^{m-3}}=1$.

$Q^{-1}PQ=Q^{2a}P^{\omega_1+2^{m-6}b_3}$, $Q^{-2}PQ^2=Q^4P^{1+2^{m-5}\beta_3}$, $Q^8=1$; $a=1$, 3; $\omega_1=1$, $m>7$, $b_3=\beta_3=\kappa=0$, 1; $m=7$, $b_3=\beta_3=\kappa=0$; $\omega_1=-1$, $b_3=4$, $\beta_3=0$, 2, $\kappa=0$; $m=7$, $b_3=\beta_3=\kappa=0$.

$Q^{-1}PQ=Q^{2a}P^{-1+2^{m-5}b_3}$, $Q^{-2}PQ^2=Q^4P^{1+2^{m-5}(\beta_3-\lambda_2)}$, $Q^8=P^{2^{m-4}\lambda_2}$; $m>7$; $\kappa=0$, $b_3=\beta_3=0$, $a=1$, 3, $\lambda_2=0$, 1; $\kappa=1$; $\beta_3=1$, $\lambda_2=0$, 1; $a=1$, $b_3=1$; $a=3$, $b_3=0$; $\beta_3=3$, $\lambda_2=1$, $a=1$, $b_3=0$, 1; $m=7$, $a=3$, $\kappa=\beta_3=1$, $\lambda=b_3=0$.

$Q^{-1}PQ=Q^{2a}P^{-1-4a_1+2^{m-6}b_3}$, $Q^{-2}PQ^2=Q^4P^{1-4-2^{m-5}\beta_3}$, $Q^8=P^8$; $\beta_3=\kappa=0$; $a=1$, 3, $b_3=0$, 2; $m>8$, $a=1$, $b_3=1$; $m=8$, $a=3$, $b_3=1$; $\beta_3=\kappa=1$; $m>7$, $a=3$, $b_3=1$, 3; $m=7$, $a=1$, $b_3=2$, 3; $a=3$, $b_3=0$, 2, 6.

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*For $m=8$, $b_3=1$, $a=3$.

†For $m=8$, $b_3-b_3'-2ax_1'[1+2a_1+2\beta_3+2b_3+2\kappa(a_1+1)]\equiv 0 \pmod 8$.

‡For $m=8$, $\beta_3=0$, $b_3=1+4b_4$; $\beta_3=2$, $b_3=3+4b_4$.

PART 2. Q^4 IS IN $\{P\}$ AND Q^2 IS NOT IN $\{P\}$.

We have $Q^4=P^{4\lambda}$. Two cases arise. They will be treated in separate sections.

§1. $\{P, Q\}$ IS OF ORDER 2^{m-1}.

Here $\{P, Q\}=G_{m-1}$ is determined from the types in III, Introduction, by replacing m by $m-1$. The types of G_{m-1} are now designated by (A), (B)... Thus writing $Q^{-1}PQ=P^{\omega+2^{m-5}\beta_1}$...(1), $Q^4=P^{2^{m-4}\lambda_1}$, we have

(A) and (B), $\omega=\pm1$, $\beta_1=0, 1, 2$, $\lambda_1=0$; (C), $\omega=-1$, $\beta_1=0$, $\lambda_1=1$; and

$Q^{-2}PQ^2=Q^{1+2^{m-4}\kappa}$ ...(2);

(D), $Q^{-1}PQ=Q^2P^{-1+2^{m-4}\beta_2}$...(1), $Q^4=1$; $\beta_2=0, 1$, $\kappa=0, 1$;

(E), $Q^{-1}PQ=Q^2P^{1-2^{m-5}\kappa}$ ...(1), $Q^4=1$, $\kappa=0, 1$;

(F), $Q^{-1}PQ=Q^2P^{-1}$...(1), $Q^4=P^{2^{m-4}}$;

(G), $Q^{-1}PQ=Q^2P^{-1+2^{m-5}\beta_2}$...(1), $Q^4=P^4$, $\beta_2=0, 1$, $\kappa=0$; $\beta_2=0$, $\kappa=1$.

Let R be an operator of G_m, not in G_{m-1}. Since R^2 is in G_{m-1}, $R^2=Q^\mu P^\nu$...(3). Then $G_m=[R, G_{m-1}]$. In G_m, $R^{-1}PR=Q^aP^b$...(4), and $R^{-1}QR=Q^cP^d$...(5). Consider R with each G_{m-1}. For (A), (B), and (C), $[0, -y, x, 0, y]=[0, 0, x(\omega+2^{m-5}\beta_1)^y]$...(6). Hence $[0, y, x]^s=[0, sy, (s-\theta_s)x\{\frac{1+(\omega)^y}{2}+(\omega)^{y-1}\times 2^{m-6}\beta_1 y\}+\theta_s x\{1+2^{m-5}\beta_1(s-1)y\}$...(7).

(A) From (3), (4), and (5), by means of (7), $b=1+2b_1$, $d=2^{m-5}d_1$, $\mu=2\mu_1$, $\nu=2\nu_1$. $(RP)^4$ is in $\{P\}$. Hence $a=2a_1$. Transforming (1), (3), (4), and (5) by R, we get $c=1+2c_1$, and the congruences

$\beta_1c_1\equiv0 \pmod 2$...(8), $2^{m-6}(d_1\mu_1+a_1\nu_1\beta_1)+b_1\nu_1\equiv0 \pmod{2^{m-5}}$...(9),

$2^{m-6}a_1d_1+b_1(1+b_1)+2^{m-6}(a_1b_1\beta_1-\beta_1\mu_1)\equiv0 \pmod{2^{m-5}}$...(10),

and $d_1(1+b_1+c_1)\equiv\beta_1\nu_1 \pmod 2$...(11).

$[z, y, x]^{2s_1}=[0, 2\{y+z(\mu_1+c_1y+a_1x)\}s_1, 2s_1\{x+2^{m-6}\beta_1xy+z(\nu_1+b_1x+2^{m-5}b_1\beta_1xy+2^{m-6}d_1y+2^{m-6}a_1\beta_1(x-\theta_x)]\}]$...(12), and

$[z, y, x]^{2s_1+1}=[z, y(1+2s_1)+2s_1z(\mu_1+c_1y+a_1x), x(1+2s_1)+2^{m-4}\beta_1s_1x\{y+z(\mu_1+c_1y+a_1x)\}+2s_1\{2^{m-6}\beta_1xy+z[\nu_1+b_1x+2^{m-5}b_1\beta_1xy+2^{m-6}d_1y+2^{m-6}a_1\beta_1(x-\theta_x)]\}]$...(13).

By means of the general correspondence $\begin{bmatrix} P', & Q', & R' \\ P, & Q, & R \end{bmatrix}$, where $P'=R^zQ^yP^x$, $Q'=R^{z'}Q^{y'}P^{x'}$, $R'=R^{z''}Q^{y''}P^{x''}$, the groups in (A) are simply isomorphic with the types given below.

(A_1) $b_1=2^{m-6}b_2$ $(b_2=0...3)$. x is odd, $x'=2^{m-6}x_1'$, $x''=2^{m-6}x_1''$, and the variables satisfy the congruences $y'+z'(\mu_1+c_1y')=k$ (k odd), $x_1'+d_1y'z'\equiv0$

(mod 2), $x_1''+d_1y''z''\equiv 0$ (mod 2), $a_1z'+c_1(yz'-y'z)\equiv 0$ (mod 2), $b_2(xz'-2^{m-6}x_1'z)+\beta_1(xy'-2^{m-6}x_1'y)+2a_1\beta_1(x_1z'-2^{m-7}x_1'z)+d_1(yz'-y'z)\equiv\beta_1'\{x+2^{m-6}(\beta_1xy+b_2xz+d_1yz)\}$ (mod 4), $a_1z''+c_1(yz''-y''z)\equiv a_1'[y'+z'(\mu_1+c_1y')]$ (mod 2), $b_2(xz''-2^{m-6}x_1''z)+\beta_1(xy''-2^{m-6}x_1''y)+2a_1\beta_1[2x_1z''-2^{m-6}x_1''z]+d_1(yz''-y''z)\equiv 2d_1a_1'z[y'+z'(\mu_1+c_1y')]+a_1'[2^{m-6}\beta_1x_1'y'+z'(d_1y'+2^{m-6}a_1\beta_1x_1')]+a_1'x_1'\times(1+2^{m-6}b_2z')+b_2'[x(1+2^{m-6}\beta_1y)+2^{m-6}z(b_2x+d_1y)]$ (mod 4), $c_1(y'z''-y''z')\equiv c_1'[y'+z'(\mu_1+c_1y')]$(mod2), $2^{m-6}\{b_2(x_1'z''-x_1''z')+\beta_1[x_1'y''-x_1''y'+a_1(x_1'z''-x_1''z')]\}+d_1(y'z''-y''z')\equiv c_1'\{x_1'[1+2^{m-6}(b_2z'+\beta_1y'+a_1\beta_1z')]+d_1y'z'\}+d_1'[x+2^{m-6}(\beta_1xy+b_2xz+d_1yz)]$ (mod 4), $y''+z''(\mu_1+c_1y'')\equiv\mu_1'[y'+z'(\mu_1+c_1y')]$ (mod 2), $x_1''+2^{m-6}\beta_1x_1''y''+z''[d_1y''+2^{m-6}x_1''(b_2+a_1\beta_1)]\equiv\mu_1'\{x_1'+2^{m-6}\beta_1x_1'y'+z'[d_1y'+2^{m-6}x_1'(b_2+a_1\beta_1)]\}$ (mod 4).

(A_2) $b_1=-1+2^{m-6}b_2$, ($b_2=0...3$). In this case x and y' are odd, $z=z'=0$, $z''=1$, $x'=2^{m-5}x_2'$, and the variables satisfy the congruences $b_2x''+\beta_1x''y''++d_1y''\equiv 0$ (mod 2), $\beta_1y'\equiv\beta_1'(1+2^{m-6}\beta_1y)$(mod 4), $a_1'\equiv a_1+y(1+c_1)$ (mod 2), $b_2x-\beta_1\{xy''+x''y-2a_1(x_1+x'')-2c_1x''y+xy\}+d_1y\equiv 2a_1'x_2'+b_2'x(1+2^{m-6}\beta_1y)$ (mod 4), $c_1\equiv c_1'$ (mod 2), $2x_2'(1+c_1)-\beta_1x''(y'-2c_1)+d_1y'\equiv d_1'x(1+2^{m-6}\beta_1y)$ (mod 4), $\mu_1+y''(1+c_1)+a_1x''\equiv\mu_1'$ (mod2), $b_2x''+d_1y''+\beta_1[a_1(x''-\theta_{x''})-x''y'']+2\nu_3\equiv 2\mu_1'x_2'+\nu_2'x(1+2^{m-6}\beta_1y)$ (mod 4). There are eighteen types in (A_1), twenty-eight in (A_2). These are tabulated below.

$$Q^{-1}PQ=P^{1+2^{m-5}\beta_1},\ R^{-1}PR=Q^{2a_1}P^{\omega_1+2^{m-5}b_2},\ R^{-1}QR=Q^{1+2c_1}P^{2^{m-5}d_1},$$

$$R^2=Q^{2\mu_1}P^{2^{m-4}\nu_2},\ Q^4=1,\ P^{2^{m-3}}=1.$$

a_1	c_1	μ_1	ν_2	b_2	d_1	β_1	ω_1	a_1	c_1	μ_1	ν_2	b_2	d_1	β_1	a_1	c_1	μ_1	ν_2	b_2	d_1	β_1
0,1	0	0	0	0	0	0,1,2	1	0	1	0	0	2	2	2	0	1	1	0	0	0	0,1
0	0,1	0	0	2	0	0	1	0	0	0	1	0	0	0,2	0,1	1	1	0	2	0	0
0	1	0	0	0	0,1	0,2	1	1	1	0	1	0	0	0,2	0	1	0	0	0	2	0
0	1	1	0	0,2	0	0	1	0	1	0	1	0	0	0	0	1	0	0	2	0	0
0,1	0	0	0	0	2	0,1	1	0,1	1	0	0	0	0	0,2	0	0	0	0	2	1	1
0	0	0	0	0	0,1	0,2	−1	1	1	0	0	0	2	0,2	0	1	1	0	0	2	0
0	0	0	0	2	0	0,2	−1	0	0	0	0	0	0	1	1	1	1	0	2	2	2
										$\omega_1=-1$							$\omega_1=-1$				

(B) From (3) and (4), by means of (7), $a=2a_1$, $b=1+2b_1$, $\mu=2\mu_1$, $\nu=2\nu_1$. Also $(RQ)^4$ is in $\{P\}$; hence $c=1+2c_1$, and $d=2d_1$. Transforming (1), (3), (4), and (5) by R, we obtain

$\beta_1(a_1+c_1)\equiv 0$ (mod 2)...(8),

$\nu_1b_1+2^{m-6}\beta_1(\mu_1d_1+a_1\nu_1)\equiv 0$ (mod 2^{m-5})...(9),

$d_1(1+b_1)-\nu_1(1-2^{m-6}\beta_1)\equiv 0$ (mod 2^{m-5})...(10), and

$b_1(1+b_1)+2^{m-6}\beta_1[a_1(b_1+d_1)+\mu_1]\equiv 0$ (mod 2^{m-5})...(11).

$[z, y, x]^{2s_1}=[0,\ 2s_1\{y+z(\mu_1+c_1y+a_1x)\},\ 2s_1\{(\phi_y+(-1)^{y-1}2^{m-6}\beta_1y)x+z[\nu_1+(-1)^y(b_1x+d_1\theta_y)(1-2^{m-5}\beta_1y)+2^{m-6}\beta_1(a_1(x-\theta_x)+d_1(y-\theta_y))]\}]$.(12),

and $[z, y, x]^{2s_1+1}=[z,\ y(1+2s_1)+2s_1z(\mu_1+c_1y+a_1x),\ x\{1-2^{m-4}\beta_1\ s_1[y+z(\mu_1+c_1y+a_1x)]\}+2s_1\{x(\phi_y+(-1)^{y-1}2^{m-6}\beta_1y)+z[\nu_1+(-1)^y(b_1x+d_1\theta_y)(1-2^{m-5}\beta_1y)+2^{m-6}\beta_1(a_1(x-\theta_x)+d_1(y-\theta_y))]\}]$...(13).

The groups in (B) for ν_1 other than 0 are isomorphic with those for $\nu_1=0$; with $C=\begin{bmatrix} P, & Q, & R' \\ P, & Q, & R \end{bmatrix}$, where $z''=1$, y'' is even for b_1 even, and odd for b_1 odd, and x'' and y'' satisfy the congruence $\nu_1+x''\theta_{y''}+(-1)^{y''}(b_1x''+d_1\theta_{y''})(1-2^{m-5}\beta_1y'')+2^{m-6}\beta_1[a_1(x''-\theta_{x''})+d_1(y''-\theta_{y''})-(-1)^{y''}x''y'']\equiv\nu_1'$ (mod 2^{m-4}).

(B_1) $b_1=2^{m-6}b_2$. From (10), $d_1=2^{m-5}d_3$.

(B_2) $b_1=-1+2^{m-6}b_2$. All the groups in (B_2) are isomorphic with those where $d_1=0, 1, 2$; with $C=\begin{bmatrix} P, & QP^{x}, & RP^{x''} \\ P, & Q, & R \end{bmatrix}$, where x' and x'' satisfy $\beta_1x'\equiv 0$ (mod 2), $\mu_1+a_1x''\equiv\mu_1'$ (mod 2), $b_2x''+\beta_1a_1(x''-\theta_{x''})\equiv\beta_1\mu_1'x'$ (mod 4), and $x''+b_1x'+d_1-2^{m-6}\beta_1[x''(1+2c_1+2a_1x')+a_1(x'-\theta_{x'})+3c_1'x']\equiv d_1'$ (mod 2^{m-4}).

The groups in (B) are simply isomorphic with the types given below, or with (A). The correspondence is given by $C=\begin{bmatrix} P', & Q', & R' \\ P, & Q, & R \end{bmatrix}$, where x and y' are odd. For (B_1), y is even. For (B_2), $y+z\equiv 0$ (mod 2); except for the groups$\sim(A)$, where for (B_1) x and y'' are odd, y and y' even, $z'=1, x'=2^{m-5}x_1'$; and for (B_2) $P'=P$, $Q'=RQP^{x'}$, $R'=R$. The variables satisfy the congruences derived in the usual way from the isomorphism. These congruences are somewhat complicated, but similar to those derived in previous cases. They are omitted for the sake of brevity.

There are in (B_1) twenty-two types; in (B_2) two for $m>7$, one for $m=7$, viz:

(B_1) $Q^{-1}PQ=P^{-1+2^{m-5}\beta_1}$, $R^{-1}PR=Q^{2a_1}P^{1+2^{m-5}b_2}$, $R^{-1}QR=Q^{1+2c_1}P^{2^{m-4}d_1}$, $R^2=Q^{2\mu_1}$, $P^{2^{m-3}}=1$.

a_1	c_1	μ_1	d_3	b_2	β_1	a_1	c_1	μ_1	d_3	b_2	β_1	a_1	c_1	ν_1	d_3	b_2	β_1
0	0	0	0	0	0,1,2	0	0	0	0	2	0	0	0	0	1	0	0,1
0	1	0	0	0,2	0,2	0	1	0	1	0	0	1	0	0	0	0,2	0,2
1	0	0	1	0,2	0	1	1	0	0,1	0	1	1	0	1	1	0,2	0

$b_2=2,\ \beta_1=2.$

(B_2) $Q^{-1}PQ=P^{-1+2^{m-5}}$, $R^{-1}PR=Q^2P^{-1+2^{m-4}b_2}$, $R^{-1}QR=Q^3P^2$, $R^2=Q^4$, $P^{2^{m-3}}=1$; $m>7$, $b_2=0, 1$; $m=7$, $b_2=0$.

(C) From (3) and (4) by means of (7), $a=2a_1$, $b=1+2b_1$, $\mu=2\mu_1$, $\nu=2\nu_1$. The transformation of (1) by R shows that $c=1+2c_1$, $a_1=0$. Also RQ must be of order 2^{m-3} at most; hence $d=2d_1$. Transforming (3), (4), and (5) by R, we obtain

$b_1\nu_1+2^{m-6}c_1\mu_1\equiv 0 \pmod{2^{m-5}}$...(8),

$b_1(1+b_1)\equiv 0 \pmod{2^{m-5}}$...(9),

$d_1(1+b_1)\equiv \nu_1 \pmod{2^{m-5}}$...(10),

$[z, y, x]^{2s_1}=[0, 2s_1\{y+z(\mu_1+c_1y)\}, 2s_1\{\phi_y x+(-1)^y z(d_1\theta_y+b_1x)+\nu_1 z\}]$...(11),

and $[z, y, x]^{2s_1+1}=[z, y(1+2s_1)+2s_1z(\mu_1+c_1y), x+2s_1\{\phi_y x+(-1)^y z(d_1\theta_y+b_1x)$
$+\nu_1 z\}]$...(12).

If $Q'^4=(RQ^{y'}P^{x'})^4=1$, and $Q'^2\neq\{P\}$, then $\mu_1+y'(1+c_1)$ is odd (13), and $x'\phi_{y'}+(-1)^{y'}(d_1\theta_{y'}+b_1x')+2^{m-6}[\mu_1+y'(1+c_1)]\equiv 0 \pmod{2^{m-5}}$. The groups satisfying these conditions do not belong in (C).

The groups for ν_1 other than 0 are isomorphic with those for $\nu_1=0$, through $C=\begin{bmatrix}P, & Q, & R' \\ P, & Q, & R\end{bmatrix}$, where $z''=1$, y'' is even for b_1 even, and odd for b_1 odd, and x'' and y'' satisfy the congruence $2^{m-5}[\mu_1+\mu_1'+y''(1+c_1)]+x''\phi_{y''}+(-1)^{y''}[d_1\theta_{y''}+b_1x'']\equiv\nu_1'-\nu_1 \pmod{2^{m-4}}$.

(C_1) $b_1=2^{m-5}b_2$. From (10), $d_1=2^{m-5}d_2$. $(C_1)\sim$the four types given below, with $C=\begin{bmatrix}P', & Q', & R' \\ P, & Q, & R\end{bmatrix}$, where x and y' are odd, y and y'' even, $z'=z=0$, $z''=1$, $x''=2^{m-5}x''$, and the variables satisfy $y_1\equiv 0 \pmod 2$, $d_2+d_2'+b_2x'+x_1''\equiv 0 \pmod 2$, $y_1''(1+c_1)-\mu_1y_1'\equiv x_1'' \pmod 2$.

(C_2) $b_1=-1+2^{m-5}b_2$. All the groups in (C_2) are isomorphic with those where $d_1=0$, by $C=\begin{bmatrix}P, & QP^{x'}, & RP^{2x_1''} \\ P, & Q, & R,\end{bmatrix}$, where x' and x_1'' satisfy $d_1'\equiv d_1+b_1x'+2x_1'' \pmod{2^{m-4}}$. Also $(C_2)\sim(C_1)$, with $C=\begin{bmatrix}P, & Q, & RQP^{2^{m-5}} \\ P, & Q, & R\end{bmatrix}$.

In (C) there are the following four types:

$$Q^{-1}PQ=P^{-1},\ R^{-1}PR=P^{1+2^{m-4}b_2}, R^{-1}QR=Q^{1+2c_1}, R^2=1,\ Q^4=P^{2^{m-4}}, P^{2^{m-3}}=1,$$
$$c_1=0, 1,\ b_2=0, 1.$$

For (D), (E), (F), and (G),

$[0, -2y_1, x, 0, 2y_1]=[0, 0, x(1+2^{m-4}\kappa y_1)]$...(6), and

$[0, 2y_1, x]^s=[0, 2sy_1, sx+2^{m-5}\kappa xy_1(s-\theta_s)]$...(7).

(D) From (1), $[0, -y, x, 0, y]=[0, 2xy, (-1)^y x+2^{m-5}\kappa\{x(y-\theta_y)+y(x-\theta_x)\}+2^{m-4}\beta_2 xy]$...(8). Hence

$[0, y, x]^s=[0, sy+(s-\theta_s)xy, \theta_s x+(s-\theta_s)\{x\phi_y+2^{m-6}\kappa[x(y-\theta_y)+y(x-\alpha_x)]+$
$2^{m-5}\beta_2 xy\}]$...(9).

From (3) and (4) by means of (9), $\mu+\nu\equiv 0 \pmod 2$, $a=2a_1$, $b=1+2b_1$. If $c=2c_1$, or if c and d were both odd, Q^2, transformed by R^2 would result in $P^{2N}=Q^2P^{2^{m-4}M}$, which is impossible. Hence $c=1+2c_1$, and $d=2d_1$. Transforming (1), (3), (4), and (5) by R, we get $\mu=2\mu_1$, $\nu=2\nu_1$, and the congruences

$\kappa(a_1+b_1+c_1+d_1)\equiv 0 \pmod 2$...(10),

$\nu_1 b_1+2^{m-6}\kappa(\mu_1 d_1+\nu_1 a_1)\equiv 0 \pmod{2^{m-5}}$...(11),

$b_1(1+b_1)+2^{m-6}\kappa[a_1(b_1+d_1)+\mu_1]\equiv 0 \pmod{2^{m-5}}$...(12),

$d_1(1+b_1)-\nu_1+2^{m-6}\kappa[d_1(a_1+c_1)+\nu_1]\equiv 0 \pmod{2^{m-3}}$...(13),

$[z,\ y,\ x]^{2s_1}=[0,\ 2s_1\{y(1+x)+z(\mu_1+c_1y+a_1x)\},\ 2s_1\{x\phi_y+2^{m-6}\kappa[x(y-\theta_y)+y(x-\theta_x)]+2^{m-5}\beta_2xy+z[\nu_1+(b_1x+d_1\theta_y)((-1)^y+2^{m-5}\kappa y)+2^{m-6}\kappa(a_1(x-\theta_x)+d_1(y-\theta_y))]\}]$...(14), and

$[z,\ y,\ x]^{2s_1+1}=[z,\ y(1+2s_1)+2s_1\{xy+z(\mu_1+c_1y+a_1x)\},\ x\{1+2^{m-4}\kappa s_1z(\mu_1+c_1y+a_1x)\}+2s_1\{x\phi_y+2^{m-6}\kappa[x(y-\theta_y)+y(x-\theta_x)]+2^{m-5}\beta_2xy+z[\nu_1+(b_1x+d_1\theta_y)((-1)^y+2^{m-5}\kappa y)+2^{m-6}\kappa(a_1(x-\theta_x)+d_1(y-\theta_y))]\}]$...(15).

The groups for ν_1 other than 0 are isomorphic with those for $\nu_1=0$ and $C=\begin{bmatrix} P, & Q, & R' \\ P, & Q, & R \end{bmatrix}$, where $z''=1$, y'' is even for b_1 even, odd for b_1 odd, and x'' and y'' satisfy the congruence $\nu_1+x''\phi_{y''}+(-1)^{y''}(b_1x''+d_1\theta_{y''})+2^{m-6}\{\kappa[(x''+d_1)(y''-\theta_{y''})+(y''+a_1)(x''-\theta_{x''})+2y''(b_1x''+d_1\theta_{y''})]+2\beta_2x''y''\}\equiv\nu_1' \pmod{2^{m-4}}$. For $\mu_1=1$, the groups where $a_1=0$ for b_1 even, and $a_1=1$ for b_1 odd, are isomorphic to (A), (B), or (C), with $C=\begin{bmatrix} P, & R, & Q^{y''}P^{x''} \\ P, & Q, & R \end{bmatrix}$, where y'' and x'' are odd and satisfy $\kappa(y_1''+x_1'')+\beta_2\equiv 0 \pmod 2$. Also the groups for $a_1=0$, b_1 odd are isomorphic to (A), (B), or (C), with $C=\begin{bmatrix} P, & R, & QP \\ P, & Q, & R \end{bmatrix}$.

(D_1) $b_1=2^{m-6}b_2$. From (13), $d_1=2^{m-5}d_2$.

(D_2) $b_1=-1+2^{m-5}b_3$. All the groups in (D_2)$\sim$those where $d_1=0, 1$, with $C=\begin{bmatrix} P, & QP^{2x_1}, & RP^{x''} \\ P, & Q, & R \end{bmatrix}$, where x_1' and x'' satisfy $\mu_1+a_1x''\equiv\mu_1' \pmod 2$, $b_2x''+2a_1\kappa x_1''\equiv 2\kappa\mu_1'x_1' \pmod 4$, $x''-2x_1'+d_1-d_1'+2^{m-5}\{x''(\beta_2+\kappa c_1)+x_1'(b_2+\kappa+\kappa c_1')\}+2^{m-6}\kappa(x''+\theta_{x''})\equiv 0 \pmod{2^{m-4}}$, $c_1-x''\equiv c_1' \pmod 2$.

The groups in (D) are simply isomorphic with the types given below, or with those in the preceding cases through $C=\begin{bmatrix} P', & Q', & R' \\ P, & Q, & R \end{bmatrix}$, x and y' are odd, x' even, $z'=0$, $z''=1$. For (D_1), y and y'' are even, and $x''=2^{m-5}x_1''$, and for (D_2), y'' is odd; except for the groups$\sim$(A), (B), or (C), where for (D_1) x is odd and y even, and for (D_2), $P'=P$, $Q'=RQP^{x'}$, $R'=R$. It has been verified that the variables, limited as above, satisfy the congruences derived in the usual way from C by means of the transforming relations of the group.

There are seventeen types in (D_1), and five in (D_2). They follow:

$$Q^{-1}PQ=Q^2P^{-1+2^{m-4}\beta_2},\quad Q^{-2}PQ^2=P^{1+2^{m-4}\kappa},\quad R^{-1}PR=Q^{2a_1}P^{\omega_1+2^{m-4}b_2},$$

$$R^{-1}QR=Q^{1+2c_2}P^{2^{m-4}d_1},\quad R^2=Q^{2\mu_1},\quad Q^4=1,\quad P^{2^{m-3}}=1.$$

a_1	c_1	μ_1	d_2	b_2	κ	β_2	ω_1	a_1	c_1	μ_1	d_2	b_2	κ	β_2	ω_1	a_1	c_1	μ_1	d_2	b_2	κ	β_1	ω_1
0	0	0	0	0	0	0,1	1	0	0	0	0	1	0	0	1	0	0	0	0,1	0	1	0	1
1	1	0	0	0	0	0,1		1	1	0	0	1	0	0		1	1	0	0,1	0	1	0	
																1	0	1	1	0	0	0	
0	0	0	1	1	0	0,1		0	0	0	1	0	0	0		0	0	0	0	0	0	0	−1
1	1	0	1	1	0	0,1		1	1	0	1	0	0	0		0	0	0	0	1	0	0,1	−1

$$Q^{-1}PQ=Q^2P^{-1+2^{m-4}\beta_2},\quad Q^{-2}PQ^2=P,\quad R^{-1}PR=P^{-1+2^{m-4}},\quad R^{-1}QR=Q^3P^2,$$

$$R^2=1,\quad Q^4=1,\quad P^{2^{m-3}}=1,\quad \beta_2=0,\ 1.$$

(*E*) From (1), $[0,\ -y,\ x,\ 0,\ y]=[0,\ 2xy,\ x+2^{m-5}\kappa\{x(y-\theta_y)-\theta_{xy}\}]$...(8), and $[0,\ y,\ x]^s=[0,\ sy+(s-\theta_s)xy,\ sx+2^{m-6}(s-\theta_s)\kappa\{x(y-\theta_y)-\theta_{xy}\}]$...(9).
From (3), (4), and (5), $\mu=2\mu_1$, $\nu=2\nu_1$, $b=1+2b_1$, $d=2^{m-5}d_1$. The operation $(RP)^4$ is in $\{P\}$, so $a=2a_1$. Transformation of (1), (3), (4), and (5) by R gives $c=1+2c_1$, and the congruences

$d_1+\kappa(b_1+c_1)\equiv 0 \pmod 2$...(10),

$b_1\nu_1+2^{m-6}(d_1\mu_1+\kappa\nu_1 a_1)\equiv 0 \pmod{2^{m-5}}$...(11),

$b_1(1+b_1)+2^{m-6}[a_1(d_1+\kappa b_1)+\kappa\mu_1]=0 \pmod{2^{m-5}}$...(12), and

$d_1(1+b_1+c_1)\equiv 0 \pmod 2$...(13). Also,

$[z,\ y,\ x]^{2s_1}=[0,\ 2s_1\{y(1+x)+z(\mu_1+c_1y+a_1x)\},\ 2s_1\{x+2^{m-6}\kappa[x(y-\theta_y)-\theta_{yx}]+$
$z[\nu_1+b_1x+2^{m-6}\kappa a_1(x-\theta_x)+2^{m-6}d_1y]\}]$...(14),

$[z,\ y,\ x]^{2s_1+1}=[z,\ \ y(1+2s_1)+2s_1\{xy+z(\mu_1+c_1y+a_1x)\},\ \ x(1+2s_1)+2s_1\{2^{m-6}\kappa$
$\times[x(y-\theta_y)-\theta_{yx}]+z[\nu_1+b_1x+2^{m-6}\kappa(a_1(x-\theta_x)+2(\mu_1+c_1y+a_1x))+$
$2^{m-6}d_1y]\}]$...(15).

(E_1) $b_1=2^{m-6}b_2$. (E_2) $b_2=-1+2^{m-6}b_2$. From (11), $\nu_1=2^{m-6}\nu_2$. The groups in (E_1) and for b_2 odd, in (E_2), for ν_1 other than 0 are isomorphic with those for $\nu_1=0$. When b_2 is even in (E_2), and $\nu_2=3$, the groups are isomorphic with those for $\nu_2=1$. $C=\begin{bmatrix}P, & Q, & R'\\ P, & Q, & R\end{bmatrix}$, where $z''=1$, and x'' and y'' satisfy $\nu_1+x''(1+b_1)+2^{m-6}\kappa[a_1(x''-\theta_{x''})+x''(y''-\theta_{y''})-\theta_{x''y''}]+2^{m-6}d_1y''\equiv\nu_1' \pmod{2^{m-4}}$. For $a_1=0$, $\mu_1=1$, $\nu_1=0,\ 2$, $(E)\sim(A)$, (B) or (C); and for $\nu_1=0$, $C_1=\begin{bmatrix}P, & R, & Q\\ P, & Q, & R\end{bmatrix}$; and for $\nu_1=2$, $C=\begin{bmatrix}RP, & Q, & R\\ P & Q, & R\end{bmatrix}$. For $a_1=1$, and $\mu_1+c_1\equiv 0 \pmod 2$, $(E)\sim(A)$, (B) or (C) with $C=\begin{bmatrix}P, & RQP^{x'}, & R\\ P, & Q, & R\end{bmatrix}$, where x' satisfies $x'(1+b_1)+2^{m-6}(\kappa x'+d_1)\equiv 0 \pmod{2^{m-5}}$. For $a_1=0,\ 1$, $\mu_1=0$, $c_1=1$, (E_1), and for

$\kappa=0$, $(E_2)\sim(A)$, (B), or (C). For (E_1), $C=\begin{bmatrix} RP, & Q, & R \\ P, & Q, & R\end{bmatrix}$; for (E_2), $a_1=0$, $C=\begin{bmatrix} RP, & QP^{2^{m-5}x_1}, & R \\ P, & Q, & R\end{bmatrix}$, where $x_1'\equiv d_2 \pmod 2$; for (E_2), $a_1=1$, $C=\begin{bmatrix} QP, & RP, & R \\ P, & Q, & R\end{bmatrix}$. For $a_1=0$, $\mu_1+c_1\equiv 0 \pmod 2$, $\kappa=0$, $(E_2)\sim(D)$ with $C=\begin{bmatrix} P, & RQ, & R \\ P, & Q, & R\end{bmatrix}$. For $\kappa=d_1=0$ in (E_1) and for $\kappa=1$ in (E), for $a_1=\mu_1=c_1=0$, the groups where $b_2=2\sim$those where $b_2=0$; with $C=\begin{bmatrix} QP, & Q, & R \\ P, & Q, & R\end{bmatrix}$, for (E_1), $\kappa=0$, and for (E_2), $\kappa=1$; and with $C=\begin{bmatrix} P^3, & Q, & RQ^2 \\ P, & Q, & R\end{bmatrix}$, for (E_1), $\kappa=1$. In (E_2), for $\kappa=1$, the groups where $a_1=0, 1$, $\mu_1=0$, $c_1=1\sim$those where $a_1=c_1=\mu_1=0$, by $C=\begin{bmatrix} P, & Q, & RQ^{y''}P \\ P, & Q, & R\end{bmatrix}$, where for $a_1=0$, $y''\equiv 2b_3' \pmod 4$, and for $a_1=1$, $y''=1+2y_1''$, and y'' satisfies $b_3+d_2+y_1''\equiv 0 \pmod 2$, and $b_3+b_3'+y_1''\equiv 0 \pmod 2$, $(b_2=1+2b_3,\ b_2'=2b_3')$. Hence there are five types in (E_1), and two in (E_2), viz:

$$Q^{-1}PQ=Q^2P^{1-2^{m-5}\kappa},\quad Q^{-2}PQ^2=P^{1+2^{m-4}\kappa},\quad R^{-1}PR=P^{\omega_1+2^{m-4}b_2},\quad R^{-1}QR=QP^{2^{m-5}d_1},\quad R^2=1,\quad Q^4=1,\quad P^{2^{m-3}}=1.$$

$\omega_1=1$; $\kappa=0, 1$, $d_1=0, 1$, $b_2=0$; $\kappa=0$, $d_1=0$, $b_2=1$; $\omega_1=-1$, $\kappa=1$, $b_2=0$, $d_1=1, 3$.

(F) From (1), $[0, -y, x, 0, y]=[1, 2x\theta_y, (-1)^y x-2^{m-5}\kappa(xy-\theta_{xy})]$...(8), and $[0, y, x]^s=[0, sy+(s-\theta_s)x\theta_y, \theta_s x+(s-\theta_s)\{x\phi_y-2^{m-6}\kappa(xy-\theta_{xy})\}]$...(9). From (3) and (4), by means of (9), $\mu+\nu\equiv 0 \pmod 2$, $a=2a_1$, $b=1+2b_1$. Transformation of (1), (3), (4), and (5) by R gives $c=1+2c_1$, $d=2d_1$, $\mu=2\mu_1$, $\nu=2\nu_1$, and the congruences

$(1+\kappa)(a_1+b_1+c_1+d_1)\equiv 0 \pmod 2$...(10),

$b_1\nu_1+2^{m-6}\{\mu_1c_1+(d_1\mu_1+a_1\nu_1)(1+\kappa)\}\equiv 0 \pmod{2^{m-5}}$...(11),

$b_1(1+b_1)+2^{m-6}\{a_1(1+b_1+c_1+d_1)+\kappa(d_1+\mu_1+a_1b_1)\}\equiv 0 \pmod{2^{m-5}}$...(12),

$d_1(1+b_1)-\nu_1+2^{m-6}\{\nu_1+d_1(a_1+c_1)\}(1+\kappa)\equiv 0 \pmod{2^{m-5}}$...(13),

$[z, y, x]^{2s_1}=[0, 2s_1\{y+x\theta_y+z(\mu_1+c_1y+a_1x)\}, 2s_1\{x\phi_y-2^{m-6}\kappa(xy-\theta_{xy})+z[\nu_1+(-1)^y(b_1x+d_1\theta_y)(1-2^{m-5}\kappa y)+2^{m-6}c_1\kappa(x-\theta_x)+2^{m-6}d_1(1+\kappa)(y-\theta_y)]\}]$ ...(14), and

$[z, y, x]^{2s_1+1}=[z, y(1+2s_1)+2s_1\{x\theta_y+z(\mu_1+c_1y+a_1x)\}, x\{1+2^{m-4}\kappa s_1[y+x\theta_y+z(\mu_1+c_1y+a_1x)]\}+2s_1\{x\phi_y-2^{m-6}\kappa(xy-\theta_{xy})+z[\nu_1+(-1)^y(b_1x+d_1\theta_y)(1-2^{m-5}\kappa y)+2^{m-6}a_1\kappa(x-\theta_x)+2^{m-6}d_1(1+\kappa)(y-\theta_y)]\}]$...(15).

If $Q'^4=(RQ^{y'}P^{x'})^4=1$, and $\{Q'^2\}\neq\{P\}$, then

$\mu_1+y'(1+c_1)+a_1x'+x'\theta_{y'}$ is odd...(16), and

$\nu_1+x'\phi_{y'}+(-1)^{y'}(b_1x'+d_1\theta_{y'})-2^{m-6}\{\kappa[y'(x'+2b_1x'+2d_1\theta_{y'})-\theta_{x'y'}]-\kappa a_1(x'-\theta_{x'})-d_1(1+\kappa)(y'-\theta_{y'})\}=2^{m-6}k$ (k odd)...(17).

The groups satisfying (16) and (17) do not belong in (F).

The groups for ν other than 0, are isomorphic with those for $\nu_1=1$, with $C=\begin{bmatrix} P, & Q, & R' \\ P, & Q, & R \end{bmatrix}$, where $z''=1$, y'' is even for b_1 even, and odd for b_1 odd, and x'' and y'' satisfy the congruence $2^{m-6}\mu_1'+\nu_1'\equiv 2^{m-6}[\mu_1+y''(1+c_1)+a_1x''+x''\theta_{y''}]+\nu_1+x''\phi_{y''}+(-1)^{y''}(b_1x''+d_1\theta_{y''})(1-2^{m-5}\kappa)-2^{m-6}\{\kappa[x''y''-\theta_{x''y''}+a_1(x''-\theta_{x''})]+d_1(1+\kappa)(y''-\theta_{y''})\} \pmod{2^{m-4}}$. For $\mu_1=1$, $a_1=0$, and for $a_1=1$, $\mu_1+c_1\equiv 0 \pmod 2$, the groups are isomorphic with (A), (B), or (C), and for $a_1=0$, $C=\begin{bmatrix} P, & R, & Q \\ P, & Q, & R \end{bmatrix}$, and for $a_1=1$, $C=\begin{bmatrix} P, & RQP^{x'}, & R \\ P, & Q, & R \end{bmatrix}$, where x' satisfies $b_1x'+d_1\equiv 0 \pmod{2^{m-5}}$.

(F_1) $b_1=2^{m-6}b_2$. From (13), $d_1=2^{m-5}d_2$.

(F_2) $b_1=-1+2^{m-6}b_2$. All the groups in (F_2)$\sim$those where $d_1=0, 1, 2$ with $C=\begin{bmatrix} P, & QP^{2x_1'}, & RP^{x''} \\ P, & Q, & R \end{bmatrix}$, where x_1' and x'' satisfy the congruences $2\mu_1x_1'(1+\kappa)-x''(a_1+b_2)+\kappa a_1(x''-\theta_{x''})\equiv 0 \pmod 4$, $d_1'-d_1+2x_1'-x''+2^{m-6}\{c_1'[1+2x_1'(1+\kappa)]-c_1(1+2\kappa x'')+\kappa(x''+\theta_{x''})+2x_1'[a_1(1+\kappa)+b_2]+x''\} \pmod{2^{m-4}}$, $c_1-x''\equiv c_1' \pmod 2$.

The groups in (F) are simply isomorphic with the types given below, or with those in preceding cases; where $Q^4=P^{2^{m-4}}$, and $C=\begin{bmatrix} P', & Q', & R' \\ P, & Q, & R \end{bmatrix}$. The variables x and y' are odd, x' and y even, $z'=0$, $z''=1$. For (F_1) y'' is even, and $x''=2^{m-5}x_1''$; for (F_2), $z=0$, and y'' is odd; except for the groups$\sim$(C), where x, y' are odd, x' even, $z=1$, $z'=0$, and for (F_1) y is even, for (F_2) y is odd. The variables specified have been proven to satisfy the congruence conditions derived from the relations of the group.

There are eight types in (F_1), and none in (F_2). The groups of (F_1) are given by

$$Q^{-1}PQ=Q^2P^{-1},\quad Q^{-2}PQ^2=P^{1+2^{m-4}\kappa},\quad R^{-1}PR=Q^{2a_1}P^{1+2^{m-4}b_2},\quad R^{-1}QR=Q^{1+2c_1}P^{2^{m-4}d_2},\quad R^2=1,\quad Q^4=P^{2^{m-4}},\quad P^{2^{m-3}}=1.$$

$\kappa=0, 1$, $b_2=0, 1$, $a_1=c_1=d_2=0$; $\kappa=0$, $b_2=0, 1$, $d_2=0, 1$, $a_1=c_1=1$.

(G) From (1),

$[0, -y, x, 0, y]=[0, 2x\theta_y, (-1)^yx+2^{m-5}\{\kappa(xy-\theta_{xy})+\beta_2x\theta_x\}]$...(8), and

$[0, y, x]^s=[0, sy+(s-\theta_s)x\theta_y, \theta_sx+(s-\theta_s)\{x\phi_y+2^{m-6}[\kappa(xy-\theta_{xy})+\beta_2x\theta_y]\}]$...(9).

From (3) and (4), by means of (9), $\mu+\nu\equiv 0 \pmod 2$, and either $a=2a_1$, $b=1+2b_1$, or $a=1+2a_1$, $b=2b_1$. For the latter, $R^{-1}PR=Q^{1+2a_1}P^{2b_1}$, from which Q may be obtained in terms of R and P. Therefore the group is generated by R and P alone, and equals $\{R, P\}$, which belongs in §2 as well as in §1.

(G_1) $a=2a_1$, $b=1+2b_1$. Transforming (1), (3), (4), and (5), and $Q^4 = P^4$ by R, we get $c=1+2c_1$, $d=2d_1$, $\mu=2\mu_1$, $\nu=2\nu_1$, and the congruences

$a_1+b_1-c_1-d_1+2^{m-6}\beta_2(a_1+d_1)\equiv 0 \pmod{2^{m-5}}$...(10),

$\mu_1(c_1+d_1)+\nu_1(a_1+b_1)+2^{m-6}[\kappa(\mu_1 d_1+\nu_1 a_1)+\beta_2\mu_1 d_1]\equiv 0 \pmod{2^{m-5}}$...(11),

$b_1(1+b_1)+a_1(1+b_1+c_1+d_1)+2^{m-6}[\kappa(a_1 b_1+a_1 d_1+\mu_1)+\beta_2 a_1 d_1]\equiv 0 \pmod{2^{m-5}}$...(12),

$d_1(1+a_1+b_1+c_1)+c_1(1+c_1)+2^{m-6}[\kappa(a_1 d_1+c_1 d_1+\nu_1)+\beta_2(c_1 d_1+\nu_1)]\equiv 0 \pmod{2^{m-5}}$...(13),

$a_1+b_1-c_1-d_1\equiv 0 \pmod{2^{m-6}}$...(14),

$[z, y, x]^{2s_1}=[0, 2s_1\{y+x\theta_y+z(\mu_1+c_1 y+a_1 x)\}, 2s_1\{x\phi_y+2^{m-6}[\kappa(xy-\theta_{xy})+\beta_2 x\theta_y] +z[\nu_1+(b_1 x+d_1 y)(1+2^{m-5}y(\kappa+\beta_2))+2^{m-6}(\kappa a_1(x-\theta_x)+d_1(\kappa+\beta_2)\times(y-\theta_y))]\}]$...(15), and

$[z, y, x]^{2s_1+1}=[z, y(1+2s_1)+2s_1\{x\theta_y+z(\mu_1+c_1 y+a_1 x)\}, x[1+2^{m-4}\kappa s_1 z(\mu_1+c_1 y +a_1 x)]+2s_1\{x\phi_y+2^{m-6}[\kappa(xy-\theta_{xy})+\beta_2 x\theta_y]+z[\nu_1+(b_2 x+d_1 y)(1+2^{m-5}y(\kappa+\beta_2))+2^{m-6}(\kappa a_1(x-\theta_x)+d_1(\kappa+\beta_2)(y-\theta_y))]\}]$...(16).

If $Q'^{2^{m-3}}=(RQ^{y'}P^{x'})^{2^{m-3}}\neq 1$, and $Q'^2\neq\{P\}$, then $\mu_1+y'(1+c_1)+x'(a_1+\theta y')=k$...(17), and $x'\phi_{y'}+b_1 x'+d_1 y'=k'$...(18) ($k$ and k' odd). The groups satisfying (17) and (18) do not belong to (G).

The groups for ν_1 other than $0\sim$those for $\nu_1=0$, with $C=\begin{bmatrix}P, & Q, & R'\\ P, & Q, & R\end{bmatrix}$ where $z''=1$, y'' is even for b_1 even, and odd for b_1 odd; and x'' and y'' satisfy the congruence $\nu_1'+\mu_1'\equiv\nu_1+\mu_1+y''(1+c_1+d_1)+x''(1+a_1+b_1)+2^{m-6}\{\kappa[a_1(x''-\theta_{x''})+x''y''-\theta_{x''y''}]+\beta_2 x''\theta_{y''}+(\kappa+\beta_2)[d_1(y''-\theta_{y''}+2y''(b_1 x''+d_1 y'')]\} \pmod{2^{m-4}}$.

(G_1) $b_1=2^{m-5}b_2$, $d_1=2^{m-5}d_2$, $a_1=c_1=\mu_1=0$. $(G_1)\sim$the five types given below and $C=\begin{bmatrix}Q, & P, & R\\ P, & Q, & R\end{bmatrix}$ and $\begin{bmatrix}P, & QP^{2^{m-6}x_1'}, & RQ^{2y_1''}P^{2x_1''}\\ P, & Q, & R\end{bmatrix}$, where x_1', x_1'' and y_1'' satisfy $x_1'+\beta_2'-\beta_2+2^{m-6}x_1'(\kappa+\beta_2)\equiv 0 \pmod 4$, $y_1''+x_1''\equiv 0 \pmod{2^{m-5}}$, $b_2'+b_2+\kappa y_1''\equiv 0 \pmod 2$, and $d_2'+d_2+(\kappa+\beta_2)x_1''\equiv 0 \pmod 2$.

(G_2) $b_1=-1+2^{m-6}b_2$, $d_1=-1+2^{m-6}(2d_3+b_2)$, $a_1=c_1=1$, $\mu_1=0$. $(G_2)\sim(G_1)$ and $C=\begin{bmatrix}P, & Q, & RQ^{y''}P\\ P, & Q, & R\end{bmatrix}$, where $y''=-1+2^{m-6}y_1''$ and satisfies $y_1''+2d_3+2^{m-6}b_2 y_1''+\beta_2(-1+2^{m-6}y_1'')\equiv 0 \pmod 4$. There are five types in (G). They are defined as follows:

$$Q^{-1}PQ=Q^2P^{-1},\quad Q^{-2}PQ^2=P^{1+2^{m-4}\kappa},\quad R^{-1}PR=P^{1+2^{m-4}b_2},\quad R^{-1}QR=QP^{2^{m-4}d_2},$$
$$R^2=1,\quad Q^4=P^4,\quad P^{2^{m-3}}=1.\quad \kappa=0;\ d_2=0,\ b_2=0,\ 1,\ b_2=d_2=1;\ \kappa=1,$$
$$d_2=0,\ b_2=0,\ 1.$$

(G') $a=1+2a_1$, $b=2b_1$. The groups for μ and ν odd $\sim$ those for μ and ν even, with $C=\begin{bmatrix} P, & Q, & RP \\ P, & Q, & R \end{bmatrix}$. Transformation of (1), (2), (3), (4), (5), and $Q^4=P^4$ by R, gives $c=2c_1$, $d=1+2d_1$, $\beta_2=0$, and the congruences

$a_1+b_1\equiv c_1+d_1 \pmod{2^{m-5}}$...(10),

$\mu_1\equiv\nu_1 \pmod{2}$...(11),

$\mu_1(c_1+d_1)+\nu_1(a_1+b_1)\equiv 0 \pmod{2^{m-5}}$...(12),

$b_1+c_1\equiv 0 \pmod{2}$...(13),

$a_1+b_1+c_1+d_1+2[a_1(b_1+c_1+d_1)+b_1^2]+2^{m-5}\kappa(a_2c_1+\mu_1)\equiv 0 \pmod{2^{m-4}}$.(14),

$a_1+b_1+c_1+d_1+2[d_1(a_1+b_1+c_1)+c_1^2]+2^{m-5}\kappa(\nu_1+b_1d_1)\equiv 0 \pmod{2^{m-4}}$ (15),

$[1, y, x]^2=[0,\ 2\mu_1+y(1+2c_1)+x(1+2a_1)+2y\theta_{x+y},\ 2\nu_1+x(1+2b_1)+(-1)^{x+y}y$
$+2d_1y+2^{m-4}(\kappa+\beta_2)\{(x+y)\nu_1+b_1xy\}+2^{m-5}\{\kappa[b_1(x-\theta_x)+c_1(y-\theta_y)+$
$y(x+y)(1+2d_1)+2a_1xy-\theta_{y(x+y)}]+\beta_2[b_1(x-\theta_x)+y\theta_{x+y}(1+2d_1)]\}]$...(16).

The groups for ν_1 other than $0\sim$ those for $\nu_1=0$, through $C=\begin{bmatrix} P, & Q, & R' \\ P, & Q, & R \end{bmatrix}$ where $z''=1$, and y'' and x'' satisfy $2\mu_1+y''(1+2c_1)+x''(1+2a_1)+2y''\theta_{x''+y''}+2\nu_1+x''(1+2b_1)+(-1)^{x''+y''}y''+2d_1y''+2^{m-4}(\kappa+\beta_2)[\nu_1(x''+y'')+b_1x''y'']+2^{m-5}\{\kappa[b_1(x''-\theta_{x''})+c_1(y''-\theta_{y''})+y''(x''+y'')(1+2d_1)+2a_1x''y''-\theta_{y''(x''+y'')}]+\beta_2[b_1(x''-\theta_{x''})+y''\theta_{x''+y''}(1+2d_1)]\}\equiv 2(\mu_1'+\nu_1') \pmod{2^{m-3}}$. The groups for $a_1+b_1\equiv 0 \pmod{2^{m-5}}\sim$ those, where $a_1=b_1=0$, with $C=\begin{bmatrix} P, & Q^{1+2a_1}P^{2b_1}, & R \\ P, & Q, & R \end{bmatrix}$.

(G_1') $b_1=2^{m-6}b_3$, $d_1=-2^{m-6}b_3$, $a_1=c_1=0$.

(G_2') $b_1=-1+2^{m-6}b_3$, $d_1=-2-2^{m-6}b_3$, $a_1=0$, $c_1=1$.

(G_3') $b_1=-2+2^{m-6}b_3$, $d_1=-1-2^{m-6}b_3$, $a_1=1$, $c_1=0$.

(G_4') $b_1=-1+2^{m-6}b_3$, $d_1=-1+2^{m-6}(b_3+2d_3)$, $a_1=c_1=1$.

(G_3') and $(G_4')\sim(G_2')$ and (G_1'), respectively, with $C=\begin{bmatrix} Q^{2y_1}P^x, & Q^{y'}Q^{2x_1'} \\ P, & Q \end{bmatrix}$ where x and y' are odd and the variables satisfy $x_1+y_1-x_1'+y_1'\equiv 0 \pmod{2^{m-5}}$, $a_1'+y_1'\equiv a_1+x_1 \pmod{2}$, $x_1+y_1-x_1'-y_1'+a_1-a_1'+b_1-b_1'+2x_1(a_1+b_1-b_1')+2y_1(c_1+d_1-b_1')-2a_1'(x_1'+y_1')+2^{m-5}\kappa\{a_1x_1'+b_1x_1+y_1(1+c_1+b_1')\}\equiv 0 \pmod{2^{m-4}}$, $c_1'+y_1\equiv c_1+x_1' \pmod{2}$, $x_1+y_1-x_1'-y_1'+c_1'-c_1+d_1'-d_1+2d_1'(x_1+y_1)+2c_1'(x_1'+y_1')-2y_1'(c_1+d_1)-2x_1'(a_1+b_1)+2^{m-5}\kappa\{c_1'x_1'+d_1'y_1+x_1'(1+b_1)+c_1y_1'\}\equiv 0 \pmod{2^{m-4}}$. The groups in (G_1') and (G_2') are simply isomorphic with the types given below, by $C=\begin{bmatrix} Q, & P, & R \\ P, & Q, & R \end{bmatrix}$, for (G_1) and for

$\kappa=0$ in (G_2); and $C=\begin{bmatrix} P, & QP^{2^{m-4}}, & R \\ P, & Q, & R \end{bmatrix}$, for $\kappa=1$, in (G_2'). There are four types in (G_1') and four in (G_2'). Their defining equations are:

$$Q^{-1}PQ=Q^2P^{-1},\ Q^{-2}PQ^2=P^{1+2^{m-4}\kappa},\ R^2=1,\ Q^4=P^4,\ P^{2^{m-3}}=1,\ \kappa=0,\ 1.$$

(G_1), $R^{-1}PR=QP^{2^{m-5}b_1}$, $R^{-1}QR=P^{1-2^{m-5}b_1}$; (G_2), $R^{-1}PR=QP^{-2+2^{m-5}b_1}$,

$R^{-1}QR=Q^2P^{-1-2^{m-5}b_1}$, $b_1=0, 1$.

§2. $\{P, Q\}$ IS OF ORDER 2^m.

$\{P, Q\}=G_m$. $\{P\}=G_{m-3}$. Two cases arise, viz: (A) Q^2 is in G_{m-2}; (B) Q^2 is not in G_{m-2}.

(A) *Q^2 is in G_{m-2}.* Here $Q^{-2}PQ^2=P^{\omega+2^{m-4}\kappa}$ ...(1), and $Q^4=P^{4\lambda}$...(2). From (1), $[0, -2y_1, x, 0, 2y_1]=[0, 0, (\omega)^{y_1}(1+2^{m-4}\kappa y_1)x]$...(3), and $[0, 2y_1, x]^s$ $=[0, 2sy_1, \{\frac{1+(\omega)^{y_1}}{2}+2^{m-5}\kappa y_1\}(s-\theta_s)x+x\theta_s]$...(4). Let R be some operator in G_{m-1}, not G_{m-2}. Then $G_{m-1}=\{R, G_{m-2}\}$. Also R^2 is in G_{m-2} and in $\{P\}$, for otherwise, $G_{m-1}=\{R, P\}=\{Q', P\}$ which has been considered in §1. Hence R^2 $=P^{2\mu}$...(5). In G_{m-1}, $R^{-1}PR=Q^{2a}P^b$...(6), $R^{-1}Q^2R=Q^{2c}P^d$...(7).

(A_1) *$\omega=1$.* From (6) by (4), $b=1+2b_1$. Now (RP^4) and $(RQ^2)^4$ are in $\{P\}$. Hence $a=0$, $c=1$, $d=2d_1$. Then $\{R, P\}$ is of order 2^{m-2} and may be reduced to five cases. $R^{-1}PR=P^{\omega_1+2^{m-4}b_2}$...(6), $R^2=P^{2^{m-4}\mu_1}$...(5). $\omega_1=\pm 1$, $b_2=0, 1$, $\mu_1=0$; $\omega_1=-1$, $b_2=0$, $\mu_1=1$. From (6), $[-y, 0, x, y]=[0, 0, (\omega_1)^y(1+2^{m-4}b_2)x]$...(8), and $[y, 0, x]^s=[sy, 0, \{\frac{1+(\omega_1)^y}{2}+2^{m-5}b_2y\}(s-\theta_s)x$ $+x\theta_s]$...(9). Transform (2) and (7) by R. There result $\lambda(1-\omega_1)+d_1\equiv 0$ (mod 2^{m-5})...(10), $d_1(1+\omega_1)\equiv 0$ (mod 2^{m-4})...(11). $[1, 2y_1, x]^2=[0, 0,$ $2^{m-4}\mu_1+4\lambda y_1+2d_1y_1+x\{\omega_1+1+2^{m-4}(\kappa y_1+b_2)\}]$...(12). In G_m, $Q^{-1}PQ=$ $RQ^{2f}P^g$...(13), $Q^{-1}RQ=RQ^{2h}P^n$ or $Q^{2h}P^n$...(14). For the second alternative of (14), the relations are inconsistent. We now transform the defining relations by Q. For $\omega_1=-1$, these results show that the relations are inconsistent. Let $Q'=QP^x$ (x odd and even). There result $d=2^{m-5}d_2$, $g=1+2g_1$, $n=2^{m-4}n_2$, $h=0$ and $\lambda=0$ except where $g_1=-1+2^{m-6}g_2$,—g_2 odd, $f=1$, g_2 odd or even, $f=0$,— when $\lambda=2^{m-6}\lambda_1$, $(\lambda_1=0, 1)$; and the congruences

$g_1=2^{m-6}g_2$, $n_2+g_2\equiv\kappa$ (mod 2)...(15); and

$g_1=-1+2^{m-6}g_2$, $\kappa f+b_2+n_2+g_2\equiv\kappa$ (mod 2)...(16).

$$[0, y, 0, z, 0, x]^s=[0, sy+(s-\theta_s)f\theta_{xy}, 0, sz+\frac{s-\theta_s}{2}xy, 0, sx+2^{m-5}s(s-1)x\{b_2xy+$$
$$\kappa(f\theta_{xy}+y)\}+(s-\theta_s)\{g_1x\theta_y+2^{m-5}[(b_2x+n_2y)z+\kappa x(\frac{y-\theta_y}{2})+y(\frac{x-\theta_x}{2})(b_2$$
$$+\kappa f+\lambda_1 f)]\}]...(17).$$

The groups in (A_1) are simply isomorphic with the types given below, with $C=\begin{bmatrix} P', & Q', & R' \\ P, & Q, & R \end{bmatrix}$, where $P'=Q^y R^z P^x$, $Q'=Q^{y'} R^{z'} P^{x'}$, $R'=Q^{2y_1''} R^{z''} P^{2^{m-5}x_1''}$. In these x and y' are odd; for g_1 even, $x'=2^{m-5}x_1'$; for g_1 odd $y=2y_1$. The variables satisfy $\kappa(1+f\theta_{x'})+\kappa'(1+g_1\theta_y)+b_2x'\equiv 0 \pmod 2$, $\kappa y_1''+z''(n_2y+b_2)\equiv b_2' \pmod 2$, $f\equiv y_1''+f'(1+f\theta_{x'}) \pmod 2$, $1\equiv z''+f'x' \pmod 2$, $2^{m-5}\{\lambda_1[f(1+x_1)+f'(1+f\theta_{x'})-y_1'']+\kappa[x'y_1+y_1'+f(x'+x_1)]+b_2[x'(1+z)+z'+x_1]+n_2[y(z'-z'')+z]+f'\theta_y[n_2z'+x_1']+g_1'[b_2z+\kappa y_1]\}-2^{m-6}x_1''+x[g_1-g_1'(1+g_1\theta_y)]\equiv 0 \pmod{2^{m-4}}$, $x'(\kappa y_1''+b_2z'')+g_2x_1''+n_2z''\equiv n_2' \pmod 2$, $x'(1+g_1)+2^{m-6}\{\lambda_1(1+fx')-\lambda_1'\}\equiv 0 \pmod{2^{m-5}}$, $\lambda_1y_1''+x_1''\equiv 0 \pmod 2$.

$(A_{\bar{1}})$ $\omega=-1$. The relations when transformed by Q are found to be inconsistent. There are therefore sixteen types in (A), viz:

$$Q^{-1}PQ=RQ^{2f}P^{\omega_2+2^{m-5}g_2},\quad Q^{-2}PQ^2=P^{1+2^{m-4}\kappa},\quad R^{-1}PR=P^{1+2^{m-4}b_2},\quad Q^{-1}RQ=RP^{2^{m-4}n_2},\quad R^2=1,\quad Q^4=P^{2^{m-4}\lambda_1},\quad P^{2^{m-3}}=1.$$

κ	b_2	f	g_2	n_2	λ_1	ω_2	κ	b_2	f	g_2	n_2	λ_1	ω_2	κ	b_2	f	g_2	n_2	λ_1	ω_2
0	0,1	0	0	0	0	1	1	0	0	0	1	0	1	1	0	0	1	0	0	−1
1	1	0	1	0	0		0	0	0	0	0	0,1	−1	0	1	0	0	1	0,1	
1	0	0	1	0	0		1	0	0	0	1	0,1		1	0	1	1	1	0	
0	0	0	1	1	0		0	0	0	1	1	0		0	1	0	1	0	0	

(B) *Q^2 is not in G_{m-2}.*

$R^{-1}PR=P^{\omega_1+2^{m-4}\kappa}$ ...(1),

$R^2=P^{2^{m-4}\mu}$ ...(2).

From (1), $[-y, 0, x, y]=[0, 0, \{(\omega_1)^y+2^{m-4}\kappa y\}x]$...(3), and

$[y, 0, x]^s=[sy, 0, \{\frac{1+(\omega_1)^y}{2}+2^{m-5}\kappa y\}(s-\theta_s)x+x\theta_s]$...(4).

In G_{m-1}, $Q^{-2}PQ^2=R^a P^b$...(5).

From (5) by (4), $b=1+2b_1$. $(Q^2P)^2=Q^4R^aP^{2b}$. If $a=1$, Q^2P would be an operator Q' of G_{m-1}, where $Q'^2\neq\{P\}$, which was discussed in §1. If $a=0$, $Q^{-2}PQ^2=P^{1+2b_1}$, and Q^2 is in $G'_{m-2}=\{Q_1^2P\}$, which was discussed in (A). Hence there are no new types in (B).

PART 3. THE SQUARE OF EVERY OPERATOR IS IN $\{P\}$.

The group G_{m-1} is determined from the types in §3, Part 2, by replacing m by $m-1$. The types of G_{m-1} are

(A) $Q^{-1}PQ=P^{1+2^{m-4}\kappa}$ ...(1), $R^{-1}PR=P^{\omega_1+2^{m-4}\beta_1}$...(2), $R^{-1}QR=QP^{2^{m-4}b_1}$

...(3), $R^2=1$, $Q^2=1$, $\omega_1=\pm 1$, $\kappa=0$, $\beta_1=0, 1$, $b_1=0$; $\beta_1=0$, $b_1=1$; $\omega_1=-1$, $\kappa=\beta_1=1$, $b_1=0, 1$.

(*B*) $Q^{-1}PQ=P$...(1), $R^{-1}PR=P^{-1}$...(2), $R^{-1}QR=Q$...(3), $R^2=P^{2^{m-4}}$, $Q^2=1$.

Let S be an operator in G_m, not in G_{m-1}.

Then $S^2=P^{2\nu}$...(4).

In $G_m=[S, G_{m-1}]$, $S^{-1}PS=R^aQ^bP^c$...(5),

$S^{-1}QS=R^eQ^fP^g$...(6),

$S^{-1}RS=R^hQ^iP^j$...(7).

$(SP)^2$, $(SQ)^2$, and $(SR)^2$ are in $[P\}$; hence $a=b=e=i=0$, $c=\omega+2^{m-4}c_1$. From (6), $g=2^{m-4}g_1$. Transformation of (7) by S shows $j=2^{m-4}j_1$. In all cases $\nu=0$, except when $c=-1$, when $\nu=2^{m-4}\nu_1$.* The groups in Part 3 are simply isomorphic with the following types:

$Q^{-1}PQ=P$, $R^{-1}PR=P^{\omega_1+2^{m-4}\beta_1}$, $R^{-1}QR=Q$, $S^{-1}PS=P^{\omega+2^{m-4}c_1}$, $S^{-1}QS=QP^{2^{m-4}g_1}$, $S^{-1}RS=RP^{2^{m-4}j_1}$, $S^2=P^{2^{m-4}\nu_1}$, $R^2=1$, $Q^2=1$, $P^{2^{m-3}}=1$.

$\omega=\omega_1=1$, $\beta_1=g_1=\nu_1=0$, $c_1=0$, $j_1=0, 1$; $c_1=1$, $j_1=0$; $\omega_1=1$, $\omega=-1$, $g_1=0$; $\nu_1=0$; $\beta_1=0$, $c_1=0$, $j_1=0, 1$; $\beta_1=0$, $c_1=1$, $j_1=0$; $\beta_1=1$, $c_1=0$, $j_1=0$; $\nu_1=1$, $\beta_1=c_1=j_1=0$; $\omega_1=-1$, $\omega=1$, $\beta_1=0, 1$, $g_1=1$, $c_1=j_1=0$.

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*Paragraph 2.